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RESEARCH ARTICLE

KINEMATICAL GENERALIZATIONS OF THE BRACHISTOCHRONE PROBLEM

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ABSTRACT

The brachistochrone problem is generalized for arbitrary velocity functions. The motion rather than the causes of the motion are considered with a kinematical analysis. In the most general form, the velocity is an arbitrary function of the independent and dependent variables as well as the slopes. Special cases of single variable velocity functions are then treated. For each case, some specific functions are considered as examples. The minimal paths are calculated using the principles of variational calculus. Curves are drawn which connects a given initial point to a final point. In some of the cases, the parametric solutions are given in the form of integrals and the constants appearing in the parametric solutions require a shooting like technique to allow the curves to reach their final destination points. The curves may be used in motion of land, marine and aerial vehicles if time is the most important factor in navigation.

KEYWORDS

Minimal Time**,** Variational Calculus, Brachistochrone problem, Vehicle Navigation, Parametric Solutions

1. Introduction

Brachistochrone is a combination of two Greek works, Brachistos meaning shortest and chrone meaning time (Deshmukh et al., 2017). The problem dates back to more than 300 years when Johan Bernoulli first solved it. The question is to find the specific curve type for which a mass descends in a minimum time under the action of gravity forces without a friction in the path in a two-dimensional vertical space. Contrary to the expectations, the path turns out to be cycloid, not linear. A detailed analytical and experimental analysis of the problem was given in a study by (Deshmukh et al., 2017). The problem was solved by employing a variety of methods such as analogy of the motion of mass to travel of light rays, a pure geometric solution, an elementary mathematical solution using the concepts of calculus, numerical solution by root-finding techniques, Bézier curve and trigonometric Bézier curve approximations, variational calculus, and solid-state physics principles (Erlichson, 1999; Boute,; Johnson, 2004; Fadzar and Misro, 2023, Hoseana et al., 2022; Deshmukh et al., 2017; Abdul-Hafidh, 2022). The discrete version of the problem was also treated in which the particle moves in a polygonal path of segments. When the segments were infinitesimally small, the cycloid solution was retrieved (Gaebler et al., 2023). The practical applications of brachistochrone curves on roller coasters, dams, water reservoirs, Hydro-power plants, curved pipes were also discussed (Ganesh et al., 2022). The dynamical generalizations of the problem include viscous or dry friction, Coulomb friction, a variable mass model incorporating linear drag force, and employment of the optimal control methods (Gladkov and Bogdanova, 2022; Šalini´, 2009; Barsuk, and Paladi, 2023; Yu Cherkasov et al., 2023; Lemak and Belousova, 2021).

In this work, instead of dealing with the causes of motion as in the case of dynamical analysis, the motion is investigated without reference to causes (forces, moments, etc.). A kinematical analysis is performed in which the velocity is assumed to be an arbitrary function of the independent, dependent variable and its derivative. The minimal time curve is determined by variational calculus. Three special sub cases are treated: 1) The velocity is an arbitrary function of the independent variable, 2) The velocity is an arbitrary function of the dependent variable, 3) The velocity is an arbitrary function of the derivative of the dependent variable. Using variational optimization, the differential equations governing the path are derived. For each case some special functional examples are treated. The solutions are expressed in the form of parametric equations. The curves starting from the origin and reaching a final point are depicted using the parametric expressions. Determining the arbitrary constants appearing in the integral form of expressions require shooting like techniques for specific curves to reach the exact final destination. The curves represent motion in a two dimensional space whether horizontal or vertical. Such minimal time curves can be employed in determining routes of land, marine and aerial vehicles if time is the major concern in the motion.

2. The General Case

The general case is treated in this section. Assume that the particle moves in a two dimensional path $y = y(x)$ with a speed function $v = v(x, y, y')$. The speed depends on the independent variable, the dependent variable and its derivative. Hence, a kinematical analysis in which the motion is investigated without reference to its causes are followed (Beer et al., 2010). The total time elapsed during the motion is

$$T = \int dt = \int \frac{ds}{v(x,y,y')} \tag{1}$$

where s is the length coordinate along the curve. From calculus (Strang, 1991) $ds = \sqrt{1+y'^2}\,dx$, and the total time is

$$T = \int \frac{\sqrt{1+y'^2}}{v(x,y,y')}\,dx \tag{2}$$

Variational calculus (O'Neil, 1991) is a powerful tool in determining the extrema of functionals. Variational calculus may also be used in determining direct integrability of differential equations (Pakdemirli, 2023). For

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$$F = \frac{\sqrt{1+y'^2}}{v(x,y,y')} \, , \tag{3}$$

taking the variation of T and equating to zero for minimal time leads to the Euler equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \, , \tag{4}$$

or upon substitution of (3) into (4) yields

$$\frac{\sqrt{1+y'^2}}{v^2}\frac{\partial v}{\partial y} + \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}\,v} - \frac{\sqrt{1+y'^2}}{v^2}\frac{\partial v}{\partial y'}\right) = 0 \, . \tag{5}$$

The above equation determines the minimal path of a particle for the prescribed speed $v(x,y,y')$. Special cases of (5) will be treated in the subsequent sections.

3. Speed Being A Function of the Independent Variable

If the speed function $v = v(x)$ solely depends on the independent variable, then from (5)

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}\,v}\right) = 0 \, , \tag{6}$$

and a first integral of the equation exists

$$\frac{y'}{\sqrt{1+y'^2}\,v} = c_1 \, . \tag{7}$$

Solving the slope

$$y' = c_1 \frac{v}{\sqrt{1-c_1^2 v^2}} \, , \tag{8}$$

and integrating

$$y = c_1 \int_0^x \frac{v}{\sqrt{1-c_1^2 v^2}} dx \, . \tag{9}$$

Therefore, the minimum paths are described by the above integral for arbitrary speeds $v = v(x)$. The initial point is selected as the origin without loss of generality, i.e., $y(0) = 0$. If the final destination point is $y(x_f) = y_f$, then c_1 has to be calculated so that the final destination can be reached by the curve. If the integral cannot be taken analytically, c_1 has to be evaluated numerically using a shooting type trial and error method. Two examples are given in this section:

i) $v = \sin x$

For this choice,

$$y = c_1 \int_0^x \frac{\sin x}{\sqrt{1-c_1^2 \sin^2 x}} dx \, . \tag{10}$$

For the final destination point of $y(2) = 1$, the coefficient is found by shooting method which is $c_1 = 0.59985$. The curve is given in Figure 1.

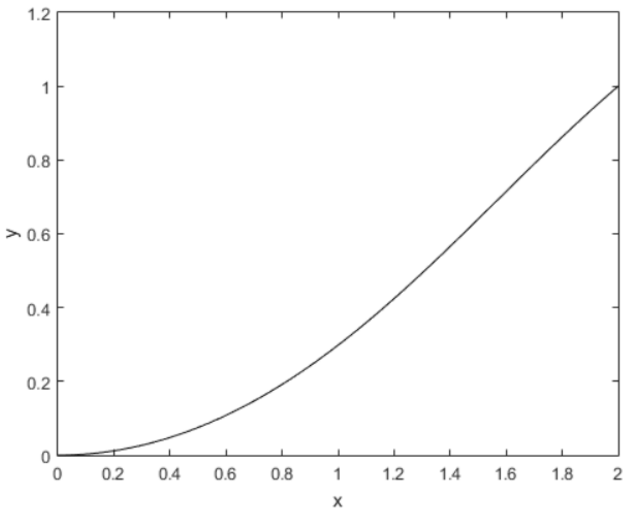

Figure 1: Minimal time curve for $v = \sin x$

ii) $v = 1 + e^{-x}$

From integral (9),

$$y = c_1 \int_0^x \frac{1+e^{-x}}{\sqrt{1-c_1^2 (1+e^{-x})^2}} dx \, . \tag{11}$$

For the final destination point of $y(2) = 1$, the integration constant is $c_1 = 0.30865$. The minimal time path is given in Figure 2.

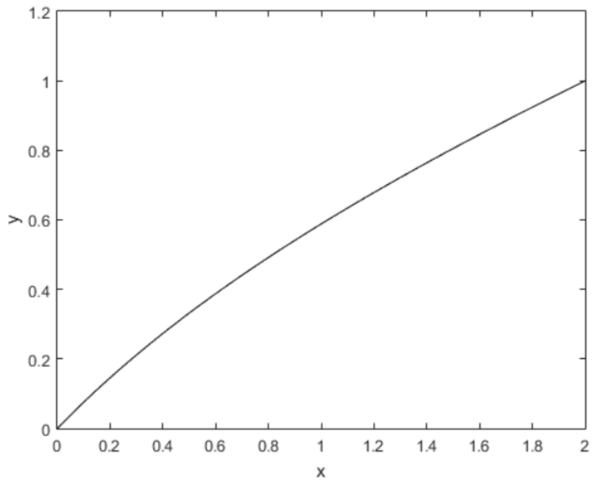

Figure 2: Minimal time curve for $v = 1 + e^{-x}$

4. Speed Being A Function of the Dependent Variable

If the speed $v = v(y)$ is a pure function of the dependent variable, then from (5)

$$\frac{\sqrt{1+y'^2}}{v^2}\frac{\partial v}{\partial y} + \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}\,v}\right) = 0\,. \quad (12)$$

Taking the derivative and making the simplifications, the final equation determining the minimal path is

$$vy'' + \frac{dv}{dy}(1 + y'^2) = 0 \quad (13)$$

Alternatively, the Euler equation (4) possesses a first integral

$$F - \frac{\partial F}{\partial y'} = \frac{1}{c_1} \quad (14)$$

for some constant c_1, if F does not explicitly depend on x (O'Neil, 1991). Substituting $F = \frac{\sqrt{1+y'^2}}{v(y)}$ into (14) and performing the algebra, the final result is

$$x = \int \frac{v}{\sqrt{c_1^2 - v^2}}dy + c_2 \quad (15)$$

One of the equations (13) or (15) can be used to determine the minimal time paths for this case. If the initial point is selected as the origin, then $c_2 = 0$, and c_1 has to determined so that the final destination point is reached. Four examples are given as special cases:

i) $v = ky$

For this choice, k being a constant, from (13), the differential equation is

$$yy'' + 1 + y'^2 = 0 \quad (16)$$

which is twice integrable (See variational theorems given in (Pakdemirli, 2023) yielding

$$y = \mp\sqrt{-x^2 + c_1 x + c_2} \quad (17)$$

which describes a circular path with center located at $(\frac{c_1}{2}, 0)$ and radius $r = \frac{c_1^2}{2} + c_2$. For the boundary conditions $y(0) = 0, y(x_f) = y_f$, the solution is

$$y = \mp\sqrt{-x^2 + \left(\frac{y_f^2}{x_f} + x_f\right)x} \quad (18)$$

For the final point of (2, -1), the circular path is given in Figure 3.

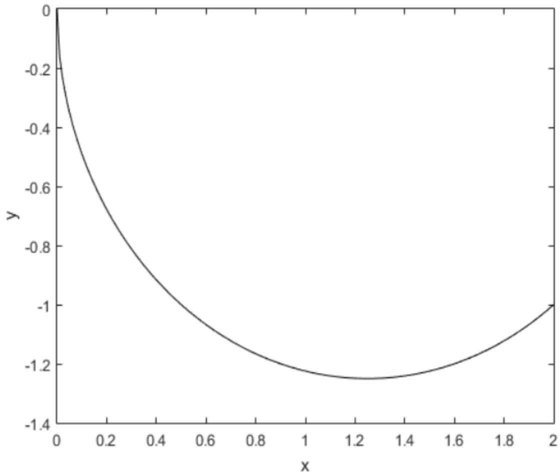

Figure 3: Minimal time curve for $v = ky$

ii) $v = k\sqrt{y}$

This case is the classical brachistochrone problem if $k = \sqrt{2g}$ where g is the gravitational acceleration. From (15),

$$x = \int \sqrt{\frac{k^2 y}{c_1^2 - k^2 y}}dy + c_2\ . \quad (19)$$

Inserting the substitution

$$y = \frac{c_1^2}{k^2}\sin^2\emptyset \qquad , \quad (20)$$

into the integral and performing the integral

$$x = \frac{c_1^2}{k^2}\left(\emptyset - \frac{1}{2}\sin 2\emptyset\right) \quad (21)$$

Defining $\theta = 2\emptyset, C = \frac{c_1^2}{2k^2}$, the parametric solution is

$$x = C(\theta - \sin\theta) + c_2 \quad (22)$$

$$y = C(1 - \cos\theta) \quad (23)$$

Applying the boundary conditions, $y(0) = 0, c_2 = 0$. For some final θ_f, the condition $y(x_f) = y_f$ is satisfied. θ_f is the root of the transcendental equation obtained by inserting x_f, y_f in (22) and (23) with $c_2 = 0$ and eliminating C between the equations

$$1 - \cos\theta_f - \frac{y_f}{x_f}(\theta_f - \sin\theta_f) = 0 \quad (24)$$

The parametric solution starting from the origin and ending at the final point (x_f, y_f) is

$$x = x_f\frac{\theta - \sin\theta}{\theta_f - \sin\theta_f} \quad (25)$$

$$y = y_f\frac{1 - \cos\theta}{1 - \cos\theta_f} \quad (26)$$

The curve is a cycloid curve. Cycloid curves can be constructed by rolling a disc and tracing the path of a point on the rim. In Figure 4, the cycloid starting from the origin and ending at (2,-1) is shown. From (24), $\theta_f = -3.5084$ by employing Newton-Raphson method.

iii) $v = ky^2$

For this choice, from (13), the differential equation is

$$yy'' + 2(1 + y'^2) = 0 \quad (27)$$

which is not directly integrable unless an integration factor is used (Pakdemirli, 2023). Expressing $y'' = y'\frac{dy'}{dy}$ and separating the variables

$$\frac{y'dy'}{1+y'^2} = -2\frac{dy}{y}. \quad (28)$$

Integrating and solving for the slopes

$$y' = \sqrt{\frac{c_1^4 - y^4}{y^4}}, \quad (29)$$

which leads to the integral

$$x = \int \sqrt{\frac{y^4}{c_1^4 - y^4}}dy + c_2 \quad (30)$$

The substitution $y^2 = c_1^2 \sin\theta$ leads to the coordinates in parametric form

$$x = \frac{c_1}{2}\int_0^\theta \sqrt{\sin\theta}d\theta + c_2 \tag{31}$$

$$y = -c_1\sqrt{\sin\theta} \tag{32}$$

If the curve starts from origin, $c_2 = 0$. Inserting $y = y_f$ in (32), solving for the constant, $c_1 = -\frac{y_f}{\sqrt{\sin\theta_f}}$ for some θ_f. Hence the curve is

$$x = -\frac{y_f}{2\sqrt{\sin\theta_f}}\int_0^\theta \sqrt{\sin\theta}d\theta \tag{33}$$

$$y = y_f\sqrt{\frac{\sin\theta}{\sin\theta_f}} \tag{34}$$

Instead of solving a transcendental equation which involves an integral, θ_f can be found by shooting method. The path is given for the final point of (2,-1) in Figure 5.

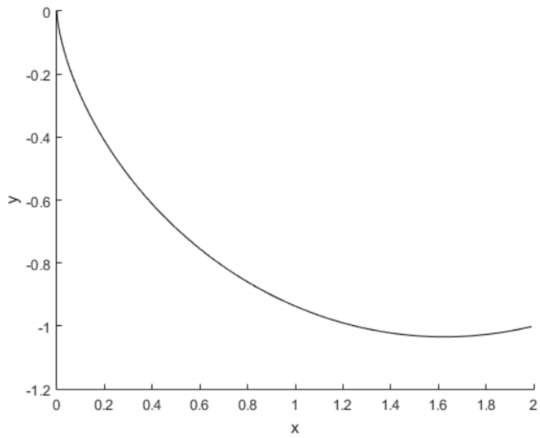

Figure 4: Minimal time curves for $v = k\sqrt{y}$

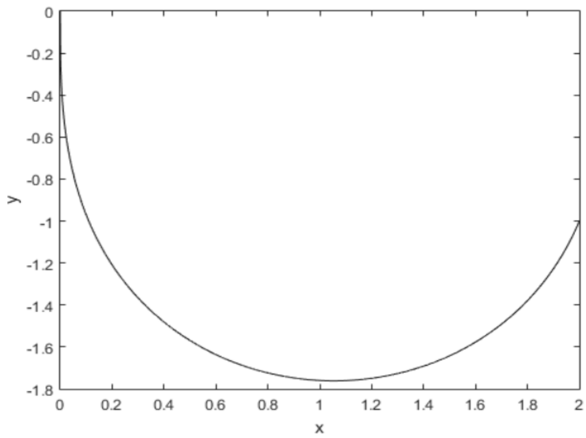

Figure 5: Minimal time curve for $v = ky^2$

For the curve, $\theta_f = 2.8133$ by the shooting method.

iv) $v = \sin y$

From (15),

$$x = \int_0^y \frac{\sin y}{\sqrt{c_1^2 - \sin^2 y}} + c_2 \tag{35}$$

The curve originates from (0,0) so, $c_2 = 0$. If the final point is (x_f, y_f), from (35)

$$x_f = \int_0^{y_f} \frac{\sin y}{\sqrt{c_1^2 - \sin^2 y}} \tag{36}$$

and the problem reduces to finding the constant c_1 within the integral. This can be done by shooting. For the end point of (0.6,1), $c_1 = 1.0134$ in figure 6.

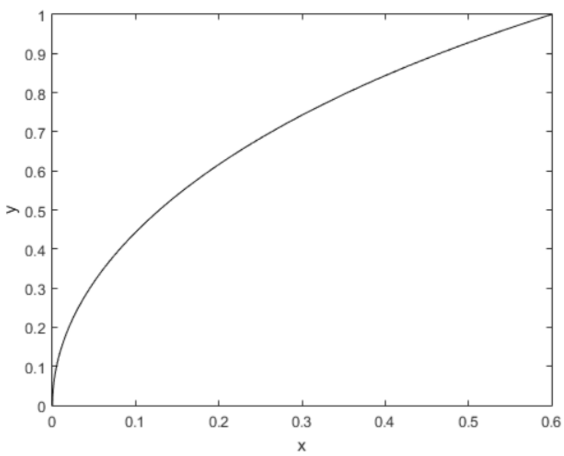

Figure 6: Minimal time curves for $v = \sin y$

5. Speed Being a Function of the Slope

If the speed function $v = v(y')$ solely depends on the slope, then from (5)

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}\,v} - \frac{\sqrt{1+y'^2}}{v^2}\frac{\partial v}{\partial y'}\right) = 0 \quad (37)$$

and a first integral of the equation is

$$\frac{y'}{\sqrt{1+y'^2}\,v} - \frac{\sqrt{1+y'^2}}{v^2}\frac{\partial v}{\partial y'} = c_1 \,. \quad (38)$$

Rearranging

$$y'v - (1+y'^2)\frac{\partial v}{\partial y'} = c_1\sqrt{1+y'^2}v^2 \,, \quad (39)$$

No matter what the functional form is, since $v = v(y')$, the equation always satisfies $y' = k_1$ for some c_1 in equation (39). Therefore, the minimum paths are always linear

$$y = k_1 x + k_2 \quad (40)$$

where $k_1$and k_2 are determined by the boundary conditions. For a path starting from the origin and reaching the point (2,-1), the linear path is

$$y = -\frac{1}{2}x \quad (41)$$

Assume that the velocity function is $v = e^{-y'}$. Then from (39)

$$y' + 1 + y'^2 = c_1\sqrt{1+y'^2}e^{-y'} \,. \quad (42)$$

For $y' = k_1 = -\frac{1}{2}, c_1 = 0.6356$. Since linear paths are always possible, this case is not technically and mathematically interesting.

6. Concluding Remarks

By considering kinematical arguments, construction of minimal time paths are outlined in this study. The general case of velocities depending on the independent and dependent variables as well as slopes is considered first. Then, special cases in which the velocity is an arbitrary function of only one of its arguments are treated. Several sub-cases one of which is the well-known brachistochrone problem is examined. Parametric solutions appear in some of the cases which may contain integrals. Starting from the origin, reaching to a final destination point requires shooting techniques or root-finding techniques as illustrated in the specific problems. Navigating vehicles usually have a control of their speed by adjusting thrust forces no matter what the external forces acting on the body is, so the analysis might be useful in predicting minimal time paths for vehicles if the most important concern is time in their motion. The analysis is valid for two dimensional curves either in a horizontal or a vertical space.

Declarations

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References

Abdul-Hafidh, E.H., 2022. A new approach to solve the Brachistochrone problem by constructing a lattice unit cell, Heliyon 8, Pp. e11994.

Barsuk, A.A., Paladi, F., 2023. On parametric representation of brachistochrone problem with Coulomb friction, International Journal of Non-Linear Mechanics, 148, 104265.

Beer, F.P., Johnston Jr, E.R., Mazurek, D.F., Cornwell, P.J., and. Eisenberg, E.R., 2010. Vector Mechanics for Engineers: Statics and Dynamics, The McGraw-Hill Co., New York.

Boute, R.T., 2012. The Brachistochrone Problem Solved Geometrically: A Very Elementary Approach, Mathematics Magazine, 85 (3), Pp. 193-199.

Cherkasov, O.Y., Malykh, E.V., and Smirnova, N.V., 2023. Brachistochrone problem and two-dimensional Goddard problem, Nonlinear Dynamics, 111, Pp. 243-254.

Deshmukh, P.C., Rajauria, P., Rajans, A., Vyshakh, B.R., and Dutta, S., 2017. The brachistochrone, Resonance, 22 (9), Pp. 847-866.

Erlichson, H., 1999. Johann Bernoulli's brachistochrone solution using Fermat's principle of least time, European Journal of Physics, 20, Pp. 299-304.

Fadzar R.A. and Misro, M.Y., 2023. Brachistochrone Curve Representation via Transition Curve, Mathematics and Statistics, 11 (1), Pp. 213-222.

Gaebler, D.J., Panaggio, M.J. and Pennings, T.J., 2023. The Discrete Brachistochrone, Mathematics Magazine, 96 (4), Pp. 376-390.

Ganesh, M., Sethuraman, N., Rangasamy, V., Navinkumar, S., and Dhinagaran, S., 2022. High efficiency hydro-powerplant with brachistochrone curve pipe, International Journal of Science Academic Research, 3 (11), Pp. 4725-4728.

Gladkov, S.O. and. Bogdanova, S.B, 2020. Analytical and numerical solution of the problem on brachistochrones in some general cases, Journal of Mathematical Sciences, 245 (4).

Hoseana, J., Kusnadi, M. R., Jeremy I. and Felisha, F., 2022. The Brachistochrone problem revisited: a numerical solution via non-linear root-finding, International Journal of Mathematical Education in Science and Technology, 53 (12), Pp. 3417-3427.

Johnson, N.P., 2004. The Brachistochrone Problem, The College Mathematics Journal, 35 (3), Pp. 192-197.

Lemak, S.S. and Belousova, M.D., 2021. The brachistochrone problem with constraints on the curvature of the trajectory, IFAC PapersOnLine 54 (13), Pp. 437-442.

O'Neil, P.V., 1991. Advanced Engineering Mathematics, Wadsworth Publishing Co., Belmont, California.

Pakdemirli, M., 2023. Application of Variational Calculus to Integrability of Differential Equations with Physical Applications, International Journal of Industrial Mathematics, 15 (2), Pp. 165-174.

Šalini´c, S., 2009. Contribution to the brachistochrone problem with Coulomb friction, Acta Mechanica, 208, Pp. 97-115.

Strang, G., 1991. Calculus, Wellesley-Cambridge Press, Wellesley.