ZIBELINE INTERNATIONAL™ PUBLISHING
ISSN: 2616-4302 (Online)
ISSN: 2990-9260 (Print)
CODEN: AMMCFL

Acta Mechanica Malaysia (AMM)

DOI: http://doi.org/10.26480/amm.01.2024.80.84

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RESEARCH ARTICLE

AREA SWEPT BY THE RADIUS OF CURVATURE OF A FUNCTION AND APPLICATIONS TO MECHANICS

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ARTICLE DETAILS

Article History:

Received 21 August 2024
Revised 24 September 2024
Accepted 18 October 2024
Available online 29 October 2024

ABSTRACT

A new integral quantity is defined in this work. For a given function in a definite interval, a special integral is defined to calculate the area swept by the radius of curvature of the function. The basic definition, theorems and examples are given first. Using variational calculus, the function corresponding to the minimum area swept by its radius of curvature is given in the form of an ordinary differential equation. The differential equation is solved and plots of the functions are given. An approximate solution is also discussed within the context and contrasted with the numerical solution. Finally, two applications from mechanics, namely the deflection of beams and the dynamics of motion in a curved path are treated.

KEYWORDS

Differential Equations, Variational Calculus, Perturbation Methods, Deflection of Beams, Kinematics

1. INTRODUCTION

The area swept by the radial distance vector of a mass found practical applications in celestial mechanics (Munson, 2023; Pourciau, 1997; Corbishley, 2000; Beer et al., 2010). According to the Kepler's second law, the radial vector of a planet moving in an elliptical orbit around sun sweeps equal areas in equal time. The area swept by the radial distance vector of a rotating spring mass was also discussed (Kenyon, 2001).

In all previous studies, the area swept by the radial vector is measured from a fixed reference point. In this study, a new area sweeping definition is given in which the area swept is measured with respect to a variable center point. The definition is about the area swept by the radius of curvature of a function in a given interval. The radius of curvature and hence the central reference point is not constant during a movement on the arbitrary function except the circular function. The basic definition, examples and theorems based on the definitions are given first. Then using the principles of variational calculus, the minimum area function is determined in the form of an ordinary differential equation which is solved analytically (O'Neil, 1991). The solution is in implicit function form. An approximate explicit solution is also given by using the perturbation theory and the solution is contrasted with the numerical solution.

Finally, in the last section, two possible areas of applications in mechanics are considered. One is the deflection of prismatic beams subject to continuous loading distribution in which the specific loading may avoid excessive stress concentrations locally if minimum area swept function is employed. The other application is to calculate the inertial normal forces acting on a body which causes side slip and turn over accidents. Employment of the minimum swept area functions may lead to smooth transitions and more even distributions in terms of the normal force components.

2. BASIC DEFINITION, THEOREMS AND EXAMPLES

The area swept by the radius of curvature can be calculated for a given function $y(x)$ in the interval $[a, b]$. The differential area is shown in Figure 1. ds is the differential length of the function at location x. The area can be approximated by a triangle with height being the radius of curvature ρ and base ds. Hence the differential area is

$$dA_\rho = \frac{1}{2}\rho ds \quad . \tag{1}$$

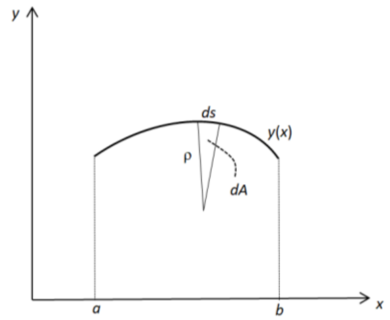

Figure 1: The differential area swept by the radius of curvature

The total area is the integral of the differential area over the interval,

$$A_\rho = \frac{1}{2}\int_a^b \rho ds \tag{2}$$

From calculus (Strang, 1991; Thomas and Finney, 1984), the absolute radius of curvature and the differential length is given by

$$\frac{1}{\rho} = \frac{|y''|}{(1+y'^2)^{3/2}}, \qquad ds = \sqrt{1+y'^2}dx \tag{3}$$

Substituting (3) into (2) yields

$$A_\rho = \frac{1}{2}\int_a^b \frac{(1+y'^2)^2}{|y''|}dx \tag{4}$$

Hence, the definition can be written as follows:

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DOI: 10.26480/amm.01.2024.80.84

Definition 1. The area swept by the radius of curvatures of a given function $y(x)$ in the interval $[a, b]$ is given by integral (4).

Example 1: To calculate the area swept by the radius of curvature of the unit circle $x^2 + y^2 = 1$ in the interval [0,1] for $y \geq 0$,

$$y = \sqrt{1-x^2}\ , \quad y' = -\frac{x}{\sqrt{1-x^2}}\ , \quad |y''| = \frac{1}{(1-x^2)^{3/2}} \tag{5}$$

Substituting (5) into (4) and simplifying yields

$$A_\rho = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \frac{1}{2}(\text{Arcsin}1 - \text{Arcsin}0) = \frac{\pi}{4} \tag{6}$$

which is the area of the quarter circle with unit radius. The radius of curvature is constant and equal to the radius for circular geometries and the area swept by the radius of curvature gives the circular area portions.

Example 2: To find the area swept by the radius of curvature of $y = 1 - e^{-x}$ in the interval [0,1], first the derivatives are calculated

$$y' = e^{-x}\ , \quad |y''| = e^{-x} \tag{7}$$

Inserting (7) into (4) and simplifying yields

$$A_\rho = \frac{1}{2}\int_0^1 (e^x + 2e^{-x} + e^{-3x})dx = \frac{1}{2}\left(\frac{4}{3} + e - 2e^{-1} - \frac{1}{3}e^{-3}\right) = 1.6496 \tag{8}$$

Example 3: To compare the area swept by a parabola to the unit circle of Example 1 with same end points, $y = -x^2 + 1$,

$$y' = -2x\ , \quad |y''| = 2 \tag{9}$$

Inserting (9) into (4) and simplifying yields

$$A_\rho = \frac{1}{2}\int_0^1 \frac{(1+4x^2)^2}{2}dx = \frac{103}{60} = 1.7167 \tag{10}$$

which is much larger than that of the unit circle since $\frac{\pi}{4} \cong 0.7854$. This is due to the more straight portions in the parabolic function which results in higher radius of curvatures and thus yields a larger area integral.

Example 4: To find the optimum α value so that the area swept by the radius of curvature of the parabola,

$$y = 1 - \left(\frac{x}{\alpha}\right)^2 \tag{11}$$

is minimum in the domain $[0, \alpha]$, note that if $\alpha \gg 1$ or $\alpha \ll 1$, the curves tend to straight curves which means the area swept by the radius of curvature becomes arbitrarily large. Hence, for a minimum value of the area swept, one can estimate before solving the problem that $\alpha \sim O(1)$. Substituting the derivatives

$$y' = -2\frac{x}{\alpha^2}\ , \quad |y''| = \frac{2}{\alpha^2} \tag{12}$$

into integral (4) and simplifying yields

$$A_\rho = \frac{1}{2}\int_0^\alpha \frac{\left(1+4\frac{x^2}{\alpha^4}\right)^2}{\frac{2}{\alpha^2}}dx = \frac{1}{4}\left(\alpha^3 + \frac{8}{3}\alpha + \frac{16}{5\alpha}\right) \tag{13}$$

Differentiating with respect to α and equating to zero,

$$\alpha^4 + \frac{8}{9}\alpha^2 - \frac{16}{15} = 0 \tag{14}$$

Solving for the positive root for real numbers

$$\alpha^2 = -\frac{4}{9} + \frac{4}{9}\sqrt{\frac{32}{5}} \tag{15}$$

or

$$\alpha = \frac{2}{3}\sqrt{-1 + \sqrt{\frac{32}{5}}} = 0.8246 \tag{16}$$

as estimated to be of $O(1)$ before attacking the problem.

The following theorems are useful:

Theorem 1: For a given function $y(x)$ in the interval $[a, b]$, if the function contains at least an incremental portion of linearity or an inflection point, then $A_\rho \to \infty$ □

Proof 1: If there is a linear part in the subinterval $[a_1, b_1] \subset [a, b]$, then for this portion of the function $y'' = 0$. This is also true for an inflection point $c_1 \epsilon [a, b]$. The numerator is a positive real number always while the denominator vanishes for such cases which makes the area integral infinity in equation (4) □

Example 5: For the cubic function $y = x^3$ in the interval $[0, 1]$,

$$A_\rho = \frac{1}{2}\int_0^1 \frac{(1+9x^4)^2}{6x}dx = \frac{1}{12}\left(\frac{9}{2} + \frac{81}{8} - \ln 0\right) = \infty \tag{17}$$

since $x = 0$ is an inflection point.

Theorem 2: For a given function $y(x)$ which satisfies the differential equation

$$y'' = \mp\frac{1}{2}(1 + y'^2)^2 \tag{18}$$

with the boundary conditions

$$y(a) = y_a\ , \quad y(b) = y_b \tag{19}$$

the area swept by the radius of curvature depends only on the end x coordinates of the function

$$A_\rho = b - a \tag{20}$$

Proof 2: If (18) is substituted into (4), $A_\rho = \frac{1}{2}\int_a^b 2dx = b - a$ □

The solution of the differential equation (18) will be discussed within the context of minimum area functions in the next section.

Theorem 3: For a given function $y(x)$ which satisfies the differential equation

$$y'|y''| - \frac{1}{2}(1 + y'^2)^2 = 0 \tag{21}$$

with the boundary conditions

$$y(a) = y_a\ , \quad y(b) = y_b \tag{22}$$

the area swept by the radius of curvature depends only on the end y coordinates of the function

$A_\rho = y(b) - y(a)$ □

Proof 3: If (21) is substituted into (4), $A_\rho = \frac{1}{2}\int_a^b 2y'dx = y(b) - y(a)$ □

3. Functions of Minimum Swept Area

To find the specific function for which the area swept by the radius of curvature is a minimum, variational calculus (O'Neil, 1991) has to be employed in search of the optimum values. Note that the integral is of the form

$$A_\rho = \frac{1}{2}\int_a^b F(y', y'')dx \tag{23}$$

where

$$F(y', y'') = \frac{(1+y'^2)^2}{|y''|} \tag{24}$$

The optimum value of the integral corresponds to the Euler Equation (O'Neil, 1991)

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \tag{25}$$

Substituting (24) into (25), the equation is

$$\frac{d}{dx}\left(\frac{4y'(1+y'^2)}{|y''|}\right) + \frac{d^2}{dx^2}\left(\frac{(1+y'^2)^2}{y''^2}\right) = 0 \tag{26}$$

The following theorem gives the minimum area swept function

Theorem 4: The function of minimum area swept by the radius of curvature satisfies the differential equation

$$y'' = c(1 + y'^2)^2 \tag{27}$$

where c is an arbitrary constant determined by the given conditions ($c \neq 0$) □

Proof 4: Substitute (27) into (26)

$$\frac{d}{dx}\left(\frac{4y'}{c(1+y'^2)}\right)+\frac{d^2}{dx^2}\left(\frac{1}{c^2(1+y'^2)^2}\right)=0 \tag{28}$$

and differentiate the second term once and use (27)

$$\frac{d}{dx}\left(\frac{4y'}{c(1+y'^2)}\right)-\frac{d}{dx}\left(\frac{4y'}{c(1+y'^2)}\right)=0 \tag{29}$$

to see that the Euler equation is satisfied identically.

Alternatively, one can take the variation of (4) after substituting (27) into (4)

$$\delta A_\rho=\frac{1}{2}\delta\int_a^b\frac{1}{c}dx=0 \tag{30}$$

which shows that the function is optimum. In fact, the function corresponds to the minimum of the integral since the maximum of the integral is always infinity □

Example 6: Find the general solution of the minimum area function

$$y''=\frac{1}{2}c_1(1+y'^2)^2 \tag{31}$$

and apply the initial conditions to determine a specific solution

$$y(0)=1\quad,\quad y'(0)=0\quad,\qquad y''(0)=-1 \tag{32}$$

Write $y''=y'\frac{dy'}{dy}$, substitute into (31) and separate similar variables on both sides of the equation

$$\frac{2y'dy'}{(1+y'^2)^2}=c_1dy \tag{33}$$

which upon integration yields

$$y'=\sqrt{-\frac{1+c_1y+c_2}{c_1y+c_2}} \tag{34}$$

Separating the variables

$$\sqrt{-\frac{c_1y+c_2}{1+c_1y+c_2}}dy=dx \tag{35}$$

The substitution $-(c_1y+c_2)=\sin^2\theta$ transforms the equation into

$$\int(1-\cos 2\theta)d\theta=-c_1dx \tag{36}$$

with an integral

$$\theta-\frac{1}{2}\sin 2\theta=-c_1x+c_3 \tag{37}$$

Since $\theta=\mathrm{Arcsin}\sqrt{-(c_1y+c_2)}$, $\sin\theta=\sqrt{-(c_1y+c_2)}$ and $\cos\theta=\sqrt{1+c_1y+c_2}$, back substitution yields the general solution in implicit form

$$\mathrm{Arcsin}\sqrt{-(c_1y+c_2)}-\sqrt{-(c_1y+c_2)}\sqrt{1+c_1y+c_2}=-c_1x+c_3 \tag{38}$$

The specific solution satisfying the initial conditions given in (32) is

$$\mathrm{Arcsin}\sqrt{2y-1}-\sqrt{2y-1}\sqrt{2(1-y)}=2x+\frac{\pi}{2} \tag{39}$$

The plot of the above function is depicted in Figure 2 together with the unit circle function satisfying the same initial conditions.

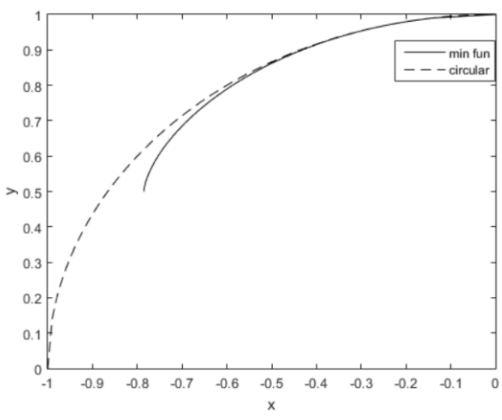

Figure 2: The minimum area function and the circular function

4. An Approximate Solution

In this section, an approximate analytical solution will be developed for the minimum swept area function

$$y''=\varepsilon(1+y'^2)^2\qquad,\quad \varepsilon\ll 1 \tag{40}$$

where ε is the small perturbation parameter. The initial conditions for the problem can be written in a more general form

$$y(0)=\alpha\quad,\quad y'(0)=\beta \tag{41}$$

Substituting the perturbation expansion

$$y=y_0+\varepsilon y_1+\varepsilon^2y_2+\cdots \tag{42}$$

into (40) and (41), and separating with respect to orders yield,

$$O(1):\quad y_0''=0,\quad y_0(0)=\alpha,\quad y_0'(0)=\beta \tag{43}$$

$$O(\varepsilon):\quad y_1''=1+y_0'^2,\quad y_1(0)=0,\quad y_1'(0)=0 \tag{44}$$

$$O(\varepsilon^2):\quad y_2''=4y_0'y_1'(1+y_0'^2),\quad y_2(0)=0,\quad y_2'(0)=0 \tag{45}$$

Solving the equations starting from the first one

$$y_0=\alpha+\beta x \tag{46}$$

$$y_1=\frac{1}{2}(1+\beta^2)^2x^2 \tag{47}$$

$$y_2=\frac{2}{3}\beta(1+\beta^2)^3x^3 \tag{48}$$

The approximate solution is then

$$y=\alpha+\beta x+\frac{1}{2}\varepsilon(1+\beta^2)^2x^2+\frac{2}{3}\varepsilon^2\beta(1+\beta^2)^3x^3+\cdots \tag{49}$$

The above approximate solution is contrasted with the numerical solution of (40) subject to the initial conditions (41). If x is not large, the match is perfect.

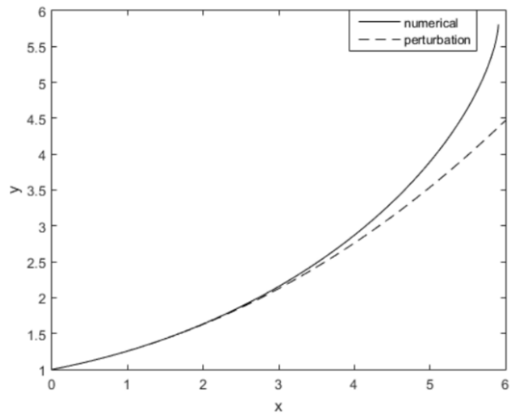

Figure 3: Comparison of the numerical and perturbation solution for the minimum area function ($\varepsilon=0.1,\alpha=1,\beta=0.2$)

5. Applications

Two applications are considered in this section. One is from the strength of materials and the other is from dynamics

5.1 Deflection of Beams

Beams are extremely important elements employed to withstand loads in static and dynamical structures. Assume a prismatic cantilever beam which is deflected transversely under the application of a continuously distributed load. The aim is to find the specific load distribution corresponding to the minimum area of swept. From strength of materials (Beer and Johnston, 1992).

$$\frac{1}{\rho}=\frac{M}{EI},\qquad \frac{dM}{dx}=V,\qquad \frac{dV}{dx}=-w(x) \tag{50}$$

where ρ is the radius of curvature, M is the moment, EI is the flexural rigidity, V is the shear force and $w(x)$ is the continuous load distribution per unit length along the horizontal length coordinate x. The radius of curvature in terms of the deflection is (Beer et al., 2010)

$$\frac{1}{\rho} = \frac{y''}{(1+y'^2)^{3/2}}, \quad (51)$$

Eliminating the shear force, moment and radius of curvature between the equations and assuming constant flexural rigidity for a prismatic bar, the load distribution is

$$w(x) = -EI\frac{d^2}{dx^2}\left(\frac{y''}{(1+y'^2)^{3/2}}\right) \quad (52)$$

It is a usual practice in mechanics of materials textbooks to approximate the equation by assuming small slopes and neglecting the small y'^2 term with respect to 1 (Beer and Johnston, 1992). Hence the load equation is given as $w(x) = -EIy^{iv}$ for constant flexural rigidity. Here, the exact nonlinear relation (52) instead of the linear approximation is used in the subsequent analysis.

The minimum swept area function satisfies equation (27) and substituting for the second derivative, the specific load distribution is

$$w(x) = -EIc\frac{d^2}{dx^2}\left(\sqrt{1+y'^2}\right) \quad (53)$$

Differentiating twice and eliminating the second derivatives at each step by employing (27), the final result is

$$w(x) = -EIc^3(1+y'^2)^{5/2}(1+4y'^2) \quad (54)$$

Solving (27) numerically subject to the conditions $y(0) = 0, y'(0) = 0$ and $c = -0.5$, the absolute dimensionless load distribution and the corresponding deflection are shown in Figure 4.

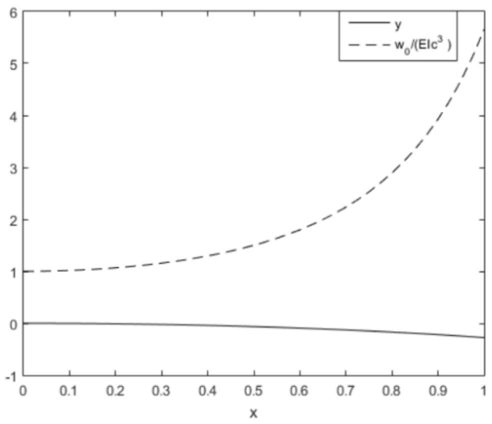

Figure 4: The minimum swept area displacement and the corresponding loading

The loading increases monotonically from the built-in support to the end of the beam. Since curvatures are more evenly distributed in a minimum swept area path, local stress concentrations are avoided in such cases which increases the chance of withstanding of the beam to excessive loading.

5.2 Normal Forces in a Path

The aim is to calculate the normal forces exerted on a vehicle in case it traces the minimum swept area path. Normal forces to the path are extremely important since they are responsible for side slip and turn over accidents in vehicles. Paths that are designed to maintain constant normal force components for prescribed tangential accelerations are investigated in detail (Pakdemirli, 2016; Pakdemirli and Dolapci, 2016; Pakdemirli, 2023; Pakdemirli and Yıldız, 2023).

For a two-dimensional plane, assume that $y(x)$ represents the curved path for a road. The normal force exerted on a moving mass is then

$$f_n = m\frac{v^2}{\rho}, \quad (55)$$

where m is the mass tracing the path and v is the constant velocity of the mass. Inserting from (51) the curvature

$$f_n = mv^2\frac{y''}{(1+y'^2)^{3/2}} \quad (56)$$

For the specific path of minimum swept area, this force can be written as

$$f_n = mv^2c\sqrt{1+y'^2} \quad (57)$$

in view of (27).

The path equation (27) is solved numerically subject to the conditions $y(0) = 0$, $y'(0) = 0$,$c = 0.6$, and substituted into (57). The path and the dimensionless normal forces exerted on the object are given in Figure 5.

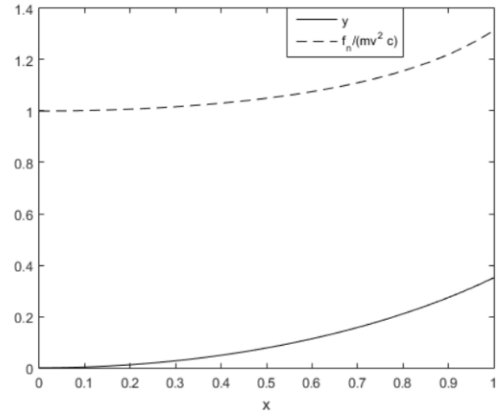

Figure 5: The minimum swept area displacement and the corresponding normal force

The normal forces exerted on the body continuously increases throughout the path but there are no abrupt changes in the normal component due to the even distribution of the curvatures in a minimum swept area path.

6. Concluding Remarks

A new integral definition is given in this work for the first time. The area swept by the radius of curvature of a function is given as the basic definition first. Examples, theorems are discussed within the context of this new definition. The differential equation yielding the minimum swept area function is derived using the variational calculus principles. The differential equations are solved analytically and numerically. Plots of minimum area functions as well as their approximations are compared.

Two applications from mechanics are given: the transverse deflection of beams and the dynamics of two dimensional planar motions in a curved path. Minimum swept area deflections may be advantageous in avoiding local stress concentrations of beams under excessive loading. On the other hand, minimum swept area paths may prevent abrupt changes in the normal force components responsible from accidents. Other application areas remain a further topic of investigation.

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